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ON THE FOUNDATIONS OF THE NONLINEAR THEORY OF THE CYLINDRICAL DEFORMATION OF THIN ELASTIC PLATES

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Abstract—Equations for the cylindrical deformation of thin plates are derived from the equations of three-dimensional nonlinear elasticity by the method of asymptotic integration. It is assumed that deflection components are of the order of the lateral dimensions of the plate and that strain components are small. Nonlinear constitutive equations are used in the model. It is found that the middle surface is inextensible in the lowest order approximation. Middle surface extensibility comes in at the next order of approximation. The range of validity of the theory is examined by considering the solution of some simple problems.

1. INTRODUCTION

The aim of this paper is to derive equations for the large deflection of thin elastic plates using the technique of asymptotic integration of the equations of nonlinear elasticity. We consider only cylindrical deformation, assuming a condition of plane strain in the plate. It is assumed that the plate is thin, that both components of deflection are of the order of the lateral dimension of the undeformed plate and that all three strain components are small.

Dimensionless displacement and stress components are expanded in a series of powers of the small dimensionless thickness of the plate. Only enough terms are retained to obtain a consistent lowest order theory. It is found that three terms must be retained in the displacement expansions and two terms in the stress expansions. The resulting system of equations can be partially integrated to obtain plate equations. It is found that the middle surface is inextensible in the lowest order displacement components and that extensibility effects come in at the next order. This result agrees with the conclusions reached by Libai and Simmonds [2] for the large deflection of beamshells. It is also pointed out by them that the boundary conditions must allow the supports to undergo a large motion in order to permit a large deformation with small strain.

In the following section we analyze the strain–displacement relations assuming a condition of plane strain. In Section 3 the equilibrium equations are analyzed assuming a plane stress condition. In Section 4 the nonlinear elastic constitutive equations are approximated on the basis of an assumption on the nature of the complementary energy function. The final section considers the solution of some simple problems using the constitutive equations for linear isotropic elasticity and for the nonlinear orthotropic behavior of paper.

We remark that the method of asymptotic integration has been used to develop the von-Karman equations of plate theory by Ciarlet [3, 4]. It has also been used by Johnson and Urbanik [1] to develop plate equations which extend the von-Karman theory to allow nonlinear constitutive behavior. In both of these theories the transverse deflection component is of the order of the plate thickness while the lateral components are one order smaller.

2. ANALYSIS OF THE STRAIN-DISPLACEMENT RELATIONS

Take the y-direction to be the direction of zero displacement. The non-zero material strain components are given by:

$$E_{xx} = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{2.1}$$

$$2E_{xz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x} + \frac{\partial u}{\partial x}\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\frac{\partial w}{\partial z} \tag{2.2}$$

$$E_{zz} = \frac{\partial w}{\partial z} + \frac{1}{2}\left(\frac{\partial u}{\partial z}\right)^2 + \frac{1}{2}\left(\frac{\partial w}{\partial z}\right)^2 \tag{2.3}$$

where u and w are displacement components and x and z material Cartesian coordinates. The coordinate direction of x is taken along the middle surface of the undeformed plate while z is in the transverse direction. Nondimensional coordinates ξ and ζ are introduced as follows:

$$x = L\xi, \qquad z = h\zeta \tag{2.4}$$

where h is the undeformed plate thickness and L the lateral plate dimension. The thinness of the plate is expressed by the requirement that dimensionless parameter $\epsilon = h/L$ be small,

$$\epsilon = h/L \ll 1. \tag{2.5}$$

Dimensionless displacement components f and g are introduced by

$$u = Lf(\xi, \zeta, \epsilon), \qquad w = Lg(\xi, \zeta, \epsilon) \tag{2.6}$$

and strain components are scaled as follows

$$\left.\begin{aligned} E_{xx} &= \epsilon\, e_{xx}(\xi, \zeta, \epsilon) \\ E_{xz} &= \epsilon\, e_{xz}(\xi, \zeta, \epsilon) \\ E_{zz} &= \epsilon\, e_{zz}(\xi, \zeta, \epsilon). \end{aligned}\right\} \tag{2.7}$$

It is assumed that functions f, g, e_{xx}, e_{xz}, e_{zz} are $O(1)$ with respect to ϵ. Hence, displacement components are assumed to be large of $O(L)$ while the strain components remain small, being $O(\epsilon)$.

Introduce (2.4), (2.6) and (2.7) into (2.1), (2.2) and (2.3) to obtain

$$\epsilon e_{xx} = \frac{\partial f}{\partial \xi} + \frac{1}{2}\left(\frac{\partial f}{\partial \xi}\right)^2 + \frac{1}{2}\left(\frac{\partial g}{\partial \xi}\right)^2 \tag{2.8}$$

$$2\epsilon^2 e_{xz} = \frac{\partial f}{\partial \zeta} + \epsilon\frac{\partial g}{\partial \xi} + \frac{\partial f}{\partial \xi}\frac{\partial f}{\partial \zeta} + \frac{\partial g}{\partial \xi}\frac{\partial g}{\partial \zeta} \tag{2.9}$$

$$\epsilon^3 e_{zz} = \epsilon\frac{\partial g}{\partial \zeta} + \frac{1}{2}\left(\frac{\partial f}{\partial \zeta}\right)^2 + \frac{1}{2}\left(\frac{\partial g}{\partial \zeta}\right)^2. \tag{2.10}$$

Next, assume that functions f and g may be expanded in an asymptotic power series in ϵ.

$$\left.\begin{aligned} f(\xi, \zeta, \epsilon) &= f_0 + \epsilon f_1 + e^2 f_2 + \cdots \\ g(\xi, \zeta, \epsilon) &\doteq g_0 + \epsilon g_1 + \epsilon^2 g_2 + \cdots \end{aligned}\right\} \tag{2.11}$$

Functions f_i and g_i depend on ξ and ζ. In order to obtain a consistent theory, it is necessary to retain at least terms up to ϵ^2. Introduce expressions (2.11) into (2.8) and equate coefficients of powers of ϵ. The first two equations obtained from (2.8) are

$$2\frac{\partial f_0}{\partial \xi} + \left(\frac{\partial f_0}{\partial \xi}\right)^2 + \left(\frac{\partial g_0}{\partial \xi}\right)^2 = 0 \tag{2.12}$$

$$e_{xx} = \frac{\partial f_1}{\partial \xi} + \frac{\partial f_0}{\partial \xi}\frac{\partial f_1}{\partial \xi} + \frac{\partial g_0}{\partial \xi}\frac{\partial g_1}{\partial \xi}. \tag{2.13}$$

The first two equations obtained from (2.9) are

$$\frac{\partial f_0}{\partial \zeta} + \frac{\partial f_0}{\partial \xi}\frac{\partial f_0}{\partial \zeta} + \frac{\partial g_0}{\partial \xi}\frac{\partial g_0}{\partial \zeta} = 0 \tag{2.14}$$

$$\frac{\partial f_1}{\partial \zeta} + \frac{\partial g_0}{\partial \xi} + \frac{\partial f_0}{\partial \xi}\frac{\partial f_1}{\partial \zeta} + \frac{\partial f_1}{\partial \xi}\frac{\partial f_0}{\partial \zeta} + \frac{\partial g_0}{\partial \xi}\frac{\partial g_1}{\partial \zeta} + \frac{\partial g_1}{\partial \xi}\frac{\partial g_0}{\partial \zeta} = 0. \tag{2.15}$$

The first three equations obtained from (2.10) are

$$\left(\frac{\partial f_0}{\partial \zeta}\right)^2 + \left(\frac{\partial g_0}{\partial \zeta}\right)^2 = 0 \tag{2.16}$$

$$\frac{\partial g_0}{\partial \zeta} + \frac{\partial f_0}{\partial \zeta}\frac{\partial f_1}{\partial \zeta} + \frac{\partial g_0}{\partial \zeta}\frac{\partial g_1}{\partial \zeta} = 0 \tag{2.17}$$

$$\frac{\partial g_1}{\partial \zeta} + \frac{1}{2}\left(\frac{\partial f_1}{\partial \zeta}\right)^2 + \frac{\partial f_0}{\partial \zeta}\frac{\partial f_2}{\partial \zeta} + \frac{1}{2}\left(\frac{\partial g_1}{\partial \zeta}\right)^2 + \frac{\partial g_0}{\partial \zeta}\frac{\partial g_2}{\partial \zeta} = 0. \tag{2.18}$$

Equation (2.16) requires that both $\partial f_0/\partial\zeta = 0$ and $\partial g_0/\partial\zeta = 0$ so that functions f_0 and g_0 are independent of ζ. We write

$$f_0 = U(\xi), \qquad g_0 = W(\xi). \tag{2.19}$$

In terms of U and W, eqn (2.12) takes the form

$$(1 + U')^2 + (W')^2 = 1 \tag{2.20}$$

which has the parametric solution

$$1 + U' = \cos\theta(\xi), \qquad W' = \sin\theta(\xi). \tag{2.21}$$

Equations (2.14) and (2.17) are satisfied identically. Equation (2.18) gives

$$2\frac{\partial g_1}{\partial \zeta} + \left(\frac{\partial f_1}{\partial \zeta}\right)^2 + \left(\frac{\partial g_1}{\partial \zeta}\right)^2 = 0. \tag{2.22}$$

This equation has the parametric solution

$$\frac{\partial f_1}{\partial \zeta} = -\sin\phi, \qquad \frac{\partial g_1}{\partial \zeta} = \cos\phi - 1. \tag{2.23}$$

Substituting from eqns (2.21) and (2.23) into eqn (2.15) reduces the latter equation to

$$\sin(\phi - \theta) = 0.$$

The roots of this equation are $\phi = \theta \pmod{2\pi}$ and $\phi = (\theta + \pi) \pmod{2\pi}$. We use the first root since it corresponds to our sign convention for curvature while the second does not. Therewith, eqns (2.23) yield

$$\left.\begin{aligned} f_1 &= -(\sin\theta)\,\zeta + f_{11}(\xi) \\ g_1 &= (\cos\theta - 1)\,\zeta + g_{11}(\xi). \end{aligned}\right\} \tag{2.24}$$

With (2.19), (2.21) and (2.24), eqn (2.13) gives the lowest order term in the strain component E_{xx}.

$$E_{xx} = \epsilon\{\cos\theta\, f'_{11} + g'_{11}\sin\theta - \theta'\zeta\}. \tag{2.25}$$

Note that the form of f_0 and g_0 given by (2.19) and (2.21) prescribe that the middle surface is inextensible in the lowest order approximation. The slope of the middle surface is $\theta(\xi)$.

Using (2.6), (2.11), (2.19), and (2.24), we find that the deformation gradients are

$$\left.\begin{aligned} F_{xx} &= 1 + \frac{\partial u}{\partial x} = \cos\theta + \epsilon(-\cos\theta\,\theta'\zeta + f'_{11}) + O(\epsilon^2) \\ F_{zz} &= 1 + \frac{\partial w}{\partial z} = \cos\theta + O(\epsilon) \\ F_{xz} &= \frac{\partial u}{\partial z} = -\sin\theta + O(\epsilon) \\ F_{zx} &= \frac{\partial w}{\partial x} = \sin\theta + \epsilon(-\sin\theta\,\theta'\zeta + g'_{11}) + O(\epsilon^2). \end{aligned}\right\} \quad (2.26)$$

Note that expressions for strain components E_{xz} and E_{zz} can be obtained from the third equation in the expansion of eqn (2.9) and the fourth equation in the expansion of eqn (2.10). These equations have not been considered in the above analysis as they are not needed in the lowest order approximation.

3. ANALYSIS OF THE EQUILIBRIUM EQUATIONS

Let the generic symbol P denote the Piola–Kirchhoff stress of the first kind and T the Piola–Kirchhoff stress of second kind. These stress components are related by

$$\left.\begin{aligned} P_{xx} &= T_{xx}F_{xx} + T_{xz}F_{xz} \\ P_{zz} &= T_{zx}F_{zx} + T_{zz}F_{zz} \\ P_{xz} &= T_{xx}F_{zx} + T_{xz}F_{zz} \\ P_{zx} &= T_{zx}F_{xx} + T_{zz}F_{xz}. \end{aligned}\right\} \quad (3.1)$$

Moment equilibrium is obtained by taking T as symmetric; force equilibrium takes the form

$$\left.\begin{aligned} \epsilon\frac{\partial P_{xx}}{\partial \xi} + \frac{\partial P_{zx}}{\partial \zeta} &= 0 \\ \epsilon\frac{\partial P_{xz}}{\partial \xi} + \frac{\partial P_{zz}}{\partial \zeta} &= 0. \end{aligned}\right\} \quad (3.2)$$

In order to keep all terms in (3.2) of the same order the following expansions of the stress components are indicated:

$$\left.\begin{aligned} T_{xx} &= \tau[T_{xx0} + \epsilon T_{xx1} + \cdots] \\ T_{xz} &= \tau\epsilon[T_{xz0} + \epsilon T_{xz1} + \cdots] \\ T_{zz} &= \tau\epsilon[T_{zz0} + \epsilon T_{zz1} + \cdots] \\ P_{xx} &= \tau[P_{xx0} + \epsilon P_{xx1} + \cdots] \\ P_{xz} &= \tau[P_{xz0} + \epsilon P_{xz1} + \cdots] \\ P_{zx} &= \tau\epsilon[P_{zx0} + \epsilon P_{zx1} + \cdots] \\ P_{zz} &= \tau\epsilon[P_{zz0} + \epsilon P_{zz1} + \cdots]. \end{aligned}\right\} \quad (3.3)$$

Here, τ has the dimension of stress and the coefficients of the expansions are independent of ϵ. Equations (3.3) indicate the order of the stress components in this theory.

Substituting (2.26), (3.1) and (3.3) into (3.2) and equating coefficients of ϵ, we obtain the equations of first order:

$$\left.\begin{aligned} \frac{\partial}{\partial \xi}(T_{xx0}\cos\theta) + \frac{\partial}{\partial \zeta}(P_{zx0}) &= 0 \\ \frac{\partial}{\partial \xi}(T_{xx0}\sin\theta) + \frac{\partial}{\partial \zeta}(P_{zz0}) &= 0 \end{aligned}\right\} \quad (3.4)$$

where

$$\left.\begin{aligned} P_{zx0} &= T_{xz0}\cos\theta - T_{zz0}\sin\theta \\ P_{zz0} &= T_{xz0}\sin\theta + T_{zz0}\cos\theta. \end{aligned}\right\} \quad (3.5)$$

The equations of second order are

$$\left.\begin{aligned} \frac{\partial}{\partial \xi}[T_{xx0}(-\cos\theta\,\theta'\zeta + f'_{11}) + T_{xx1}\cos\theta - T_{xz0}\sin\theta] + \frac{\partial}{\partial \zeta}(P_{zx1}) &= 0 \\ \frac{\partial}{\partial \xi}[T_{xx0}(-\sin\theta\,\theta'\zeta + g'_{11}) + T_{xx1}\sin\theta + T_{xz0}\cos\theta] + \frac{\partial}{\partial \zeta}(P_{zz1}) &= 0. \end{aligned}\right\} \quad (3.6)$$

Next, define stress resultants and moments by

$$\left.\begin{aligned} N_{xxi} &= \int_{-1/2}^{1/2} T_{xxi}\,d\zeta,\; N_{xzi} = \int_{-1/2}^{1/2} T_{xzi}\,d\zeta \\ M_{xx0} &= \int_{-1/2}^{1/2} T_{xx0}\,\zeta\,d\zeta. \end{aligned}\right\} \quad (3.7)$$

We assume that the surface $\zeta = -\frac{1}{2}$ is stress free and that the surface $\zeta = +\frac{1}{2}$ is acted on by dead load components.

Expand these surface dead load components as follows:

$$\begin{aligned} P_{zx}(\xi, +\tfrac{1}{2}) &= \tau\epsilon[r_0(\xi) + \epsilon r_1(\xi) + \cdots] \\ P_{zz} = (\xi, +\tfrac{1}{2}) &= \tau\epsilon[q_0(\xi) + \epsilon q_1(\xi) + \cdots]. \end{aligned} \quad (3.8)$$

Plate force equilibrium equations are obtained by integrating eqns (3.4) and (3.6) with respect to ζ.

$$\frac{d}{d\xi}[N_{xx0}\cos\theta] + r_0(\xi) = 0 \quad (3.9)$$

$$\frac{d}{d\xi}[N_{xx0}\sin\theta] + q_0(\xi) = 0 \quad (3.10)$$

$$\frac{d}{d\xi}[-M_{xx0}\cos\theta\,\theta' + N_{xx0}f'_{11} + N_{xx1}\cos\theta - N_{xz0}\sin\theta] + r_1(\xi) = 0 \quad (3.11)$$

$$\frac{d}{d\xi}[-M_{xx0}\sin\theta\,\theta' + N_{xx0}g'_{11} + N_{xx1}\sin\theta + N_{xz0}\cos\theta] + q_1(\xi) = 0. \quad (3.12)$$

Plate moment equilibrium equations are obtained by integrating ζ times eqns (3.4) with respect to ζ,

$$\frac{d}{d\xi}[M_{xx0}\cos\theta] - N_{xz0}\cos\theta + N_{zz0}\sin\theta + \tfrac{1}{2}r_0(\xi) = 0 \quad (3.13)$$

$$\frac{d}{d\xi}[M_{xx0}\sin\theta] - N_{xz0}\sin\theta - N_{zz0}\cos\theta + \tfrac{1}{2}q_0(\xi) = 0. \quad (3.14)$$

In the next section, it will be shown that N_{xx0} and M_{xx0} are determined by constitutive equations as functions of f_{11}, g_{11} and θ. Hence, eqns (3.9)–(3.14) are six equations for f_{11}, g_{11}, θ, N_{xx1}, N_{xz0}, N_{zz0}. These equations are to be solved subject to appropriate boundary conditions.

Note that the first order stress components are given by

$$\left.\begin{aligned} P_{xx0} &= T_{xx0}\cos\theta, \quad P_{xz0} = T_{xx0}\sin\theta \\ P_{zx0} &= -\int_{-1/2}^{\zeta} \frac{\partial}{\partial\xi}(T_{xx0}\cos\theta)\,d\zeta \\ P_{zz0} &= -\int_{-1/2}^{\zeta} \frac{\partial}{\partial\xi}(T_{xx0}\sin\theta)\,d\zeta. \end{aligned}\right\} \tag{3.15}$$

The last two expressions are obtained by integrating eqns (3.4).

To reduce the number of equations we integrate (3.11) and (3.12) and eliminate variable N_{xx1} between them. Assume that the plate occupies the domain $0 \le \xi < 1$.

$$N_{xz0} = N_{xx0}(f'_{11}\sin\theta - g'_{11}\cos\theta) + \sin\theta\int_0^{\xi} r_1(\xi)\,d\xi$$
$$- \cos\theta\int_0^{\xi} q_1(\xi)\,d\xi - P_1\sin\theta + Q_1\cos\theta. \tag{3.16}$$

Constants P_1 and Q_1 are the values of $\int_{-1/2}^{1/2} P_{xx1}\,d\zeta$ and $\int_{-1/2}^{1/2} P_{xz1}\,d\zeta$ at $\xi = 0$. We can also eliminate N_{zz0} between eqns (3.13) and (3.14) to obtain

$$M'_{xx0} - N_{xz0} + \tfrac{1}{2}r_0\cos\theta + \tfrac{1}{2}q_0\sin\theta = 0. \tag{3.17}$$

Equations (3.9), (3.10), (3.16) and (3.17) are four equations for f_{11}, g_{11}, θ and N_{xz0}.

4. PLATE CONSTITUTIVE EQUATIONS

In the above treatment it has been assumed that $T_{xy} = T_{yz} = 0$. The P-stress components not exhibited in eqn (3.1) are zero except for $P_{yy} = T_{yy}$. The equilibrium equation omitted from eqn (3.2) is identically satisfied. These assumptions are appropriate for an isotropic material or for an orthtropic material where the coordinate directions are also directions of material symmetry.

With $E_{xy} = E_{yz} = 0$, the strain energy density is a function of the four strain components E_{xx}, E_{yy}, E_{zz}, E_{xz}. The stress–strain relations are

$$T_{xx} = \frac{\partial H}{\partial E_{xx}}, \qquad T_{yy} = \frac{\partial H}{\partial E_{yy}}, \qquad T_{zz} = \frac{\partial H}{\partial E_{zz}}, \qquad T_{xz} = \frac{\partial H}{\partial E_{xz}}. \tag{4.1}$$

Assume that relations (4.1) can be solved for strain components in terms of stress components and define the complementary energy by

$$H_C(T_{xx}, T_{yy}, T_{zz}, T_{xz})$$
$$= T_{xx}E_{xx} + T_{yy}E_{yy} + T_{zz}E_{zz} + T_{xz}E_{xz} - H(E_{xx}, E_{yy}, E_{zz}, E_{xz}). \tag{4.2}$$

From (3.3) we see that the order of the stress components is given by

$$T_{xx} = O(\tau), \qquad T_{xz} = O(\tau\epsilon), \qquad T_{zz} = O(\tau\epsilon). \tag{4.3}$$

We also assume that $T_{yy} = O(\tau)$.

One would like to approximate the strain energy function by neglecting dependence on all strain components except E_{xx} [2]. However, since all of the strain components are of the same order [eqn (2.7)], this approximation would seem to be hard to justify on the basis of a direct order of magnitude argument. In fact, it is not possible even for the linear elastic material. However, it is possible to approximate the complementary energy function in a rational manner. Consistent with the orders given in eqn (4.3), we assume that stress components T_{xz} and T_{zz} may be neglected in the complementary energy function to obtain

$$H_C \sim \tilde{H}_C(T_{xx}, T_{yy}) = H_C(T_{xx}, T_{yy}, 0, 0). \tag{4.4}$$

This represents an assumption about the nature of the material of the plate. The complementary energy must be of a form such that approximation (4.4) is valid. We note that it is valid for linear elasticity.

Stain–displacement relations are given by

$$E_{xx} \sim \frac{\partial \tilde{H}_C}{\partial T_{xx}}, \qquad E_{yy} \sim \frac{\partial \tilde{H}_C}{\partial T_{yy}} = 0. \tag{4.5}$$

Assume the second equation in (4.5) is solvable for T_{yy} in terms of T_{xx} and use this relation to eliminate T_{yy} from (4.4). We obtain

$$H_C \sim \tilde{H}_C[T_{xx}, T_{yy}(T_{xx})] = \overline{H}_C(T_{xx}) \tag{4.6}$$

$$E_{xx} \sim \frac{\partial \overline{H}_C}{\partial T_{xx}}. \tag{4.7}$$

Next, assume that (4.7) can be inverted to obtain T_{xx} in terms of E_{xx}. Then, using (2.7) and (4.3), we see from (4.2) that the strain energy can be approximated by

$$H \approx \overline{H}(E_{xx}) = T_{xx}(E_{xx})E_{xx} - \overline{H}_C[T_{xx}(E_{xx})].$$

Therewith, the plate constitutive equation is

$$T_{xx} = \frac{\partial \overline{H}(E_{xx})}{\partial E_{xx}}. \tag{4.8}$$

Note that E_{xx} is given as a function of f_{11}', g_{11}', and θ' by eqn (2.25). Hence, eqn (4.8) expresses T_{xx} as a function of these same variables and is to be used in the equilibrium equations (3.9)–(3.17).

We note for linear isotropic elasticity the above approximation procedure can be carried out and gives

$$T_{xx} = \frac{E}{1 - \nu^2} E_{xx} \tag{4.9}$$

where E is Young's modulus and ν Poisson's ratio. We see that the stress magnitude is $\tau = \epsilon E/(1 - \nu^2)$ and

$$T_{xx0} = \cos\theta\, f_{11}' + \sin\theta\, g_{11}' - \theta'. \tag{4.10}$$

The stress resultant N_{xx0} and moment resultant M_{xx0} are given by

$$N_{xx0} = \cos\theta\, f_{11}' + \sin\theta\, g_{11}', \quad M_{xx0} = -\frac{1}{12}\theta' \tag{4.11}$$

which are to be used in eqns (3.9), (3.10), (3.16), and (3.17).

For paper the strain energy function is given by [1]

$$\bar{H}(e) = (c_1^2/c_2)\log\cosh\left[\frac{c_2}{c_1}\sqrt{\left(\frac{\nu_2 e}{1-\nu_1\nu_2}\right)}\right] \tag{4.12}$$

where

$$e = \nu_2^{-1}E_{xx}^2 \tag{4.13}$$

for cylindrical deformation. c_1, c_2, ν_1, and ν_2 are constants, ν_1 and ν_2 being the orthotropic Poisson ratios, c_2 the initial Young's modulus and c_1 the ultimate stress in the x-direction. [In writing (4.12) and (4.13), the x and y directions have been interchanged from those used in [1].] Actually, (4.12) was shown in [1] to hold where E_{11} is the middle surface strain. We here assume it is valid for E_{11} given by eqn (2.25). From (4.8) and (4.12), we obtain

$$\tau = \epsilon c_1/\sqrt{(1-\nu_1\nu_2)} \quad \text{and} \quad T_{xx0} = \frac{1}{\epsilon}\tanh\left[\frac{c_2}{c_1\sqrt{(1-\nu_1\nu_2)}}E_{xx}\right]. \tag{4.14}$$

Integrating (4.14) leads to

$$N_{xx0} = \frac{1}{\epsilon a\theta'}\log\left[\frac{\cosh(e_x+\frac{1}{2}a\theta')}{\cosh(e_x-\frac{1}{2}a\theta')}\right] \tag{4.15}$$

$$M_{xx0} = \frac{1}{\epsilon}\int_{-1/2}^{1/2}\tanh(e_x - a\theta'\zeta)\zeta\,d\zeta \tag{4.16}$$

where

$$e_x = \frac{c_2\epsilon}{c_1\sqrt{(1-\nu_1\nu_2)}}(\cos\theta\, f_{11}' + \sin\theta\, g_{11}'), \qquad a = \frac{c_2\epsilon}{c_1\sqrt{(1-\nu_1\nu_2)}}. \tag{4.17}$$

5. THE "ELASTICA" AND DEAD WEIGHT PROBLEMS

We next apply these results to the problem where only an axial load is applied at $\xi = 0$ and $\xi = 1$. We take

$$h\int_{-1/2}^{1/2}P_{xx}\,d\zeta = -\bar{P}, \qquad \int_{-1/2}^{1/2}P_{xz}\,d\zeta = 0 \quad \text{at } \xi = 0, 1. \tag{5.1}$$

We also take $r_0 = r_1 = q_0 = q_1 = 0$. Equation (3.9) and (3.10) then give $N_{xx0} = 0$. This leads us to set $h\tau\epsilon P_1 = -\bar{P}$ and $Q_1 = 0$ in eqn (3.16) which then gives $N_{xz0} = -P_1\sin\theta$. Equation (3.17) becomes

$$M_{xx0}' + P_1\sin\theta = 0. \tag{5.2}$$

Equation (5.2) is valid for any material properties. Note that middle surface extensibility effects do not enter at this level of approximation.

For the linear elastic material eqn (4.11) yields

$$\theta'' - 12P_1\sin\theta = 0. \tag{5.3}$$

When this equation is written in terms of the original variables, it becomes

$$\frac{d^2\theta}{dx^2} + \frac{12(1 - \nu^2)}{Eh^3}\bar{P}\sin\theta = 0 \tag{5.4}$$

which is an equation of the same form as the equation for the elastica, as it should be.

For the paper constitutive equation (4.15), $N_{xx0} = 0$ gives $e_x = 0$ just as for the linear elastic material. Equation (4.16) then yields the following equation for θ

$$\theta'' + \frac{P_1 \sin\theta}{\dfrac{a}{\epsilon}\displaystyle\int_{-1/2}^{1/2} \frac{\zeta^2 \, d\zeta}{\cosh^2(a\theta'\zeta)}} = 0 \tag{5.5}$$

where we have replaced P_1 by $-P_1$. This equation is to be solved subject to the boundary conditions $\theta'(0) = \theta'(1) = 0$. The solution of the linearized version of this problem yields the critical buckling load $P_{cr} = a\pi^2/12\epsilon$ for P_1. The eigenfunction is

$$\theta = \theta_0 \cos(\pi\xi). \tag{5.6}$$

A standard perturbation analysis yields the postbuckling solution of (5.5) in the form (5.6) with P_1 and θ_0 related by

$$\frac{P_1}{P_{cr}} = 1 + \left[\frac{1}{8} - \frac{3\pi^2 a^2}{80}\right]\theta_0^2. \tag{5.7}$$

This result is valid for small θ_0.

For a plate loaded by its own transverse dead weight, it can be shown that the dead weight is of the same order as the q_0 term in eqn (3.8). We thus set $q_1 = r_0 = r_1 = 0$. Take the domain of the plate as $0 < \xi < 1$. Equation (3.9) and (3.10) yield

$$\left.\begin{aligned} N_{xx0}\cos\theta &= P_0 \\ N_{xx0}\sin\theta &= Q_0 - q_0\xi \end{aligned}\right\} \tag{5.8}$$

where P_0 is the dimensionless constant horizontal force in the plate and Q_0 the dimensionless vertical force at $\xi = 0$. If the plate is supported at its ends, overall force equilibrium gives $Q_0 = q_0/2$. Equations (5.8) yield

$$N_{xx0} = P_0/\cos\theta \tag{5.9}$$

$$\theta = \tan^{-1}\left[\frac{q_0}{P_0}(\tfrac{1}{2} - \xi)\right]. \tag{5.10}$$

The lowest order displacement components, U and W, are obtained by integrating eqns (2.21). Note that the plate is inextensional and the solution (5.10) is determined from the equilibrium equations alone in the lowest approximation. This would seem to correspond to the "membrane-inextensional theory" discussed by Libai and Simmonds [2]. Middle surface extension comes into play at the next higher level of approximation. Equations for f_{11} and q_{11} are obtained from eqns (3.16) and (3.17) with θ given by (5.10) and N_{xx0} and M_{xx0} given by the appropriate constitutive equations.

In the above solution, it is assumed that the plate is supported at its ends so that it is capable of carrying the load P_0. In general, it is not possible to satisfy the simple support boundary condition of zero moment at $\xi = 0, 1$. Note that for a dead weight loaded cantilevered plate, we must have $P_0 = 0$ and $Q_0 = q_0$ with which (5.8) yields $\theta = \pi/2$ and $N_{xx0} = q_0(1 - \xi)$. It is not possible to satisfy the fixed boundary condition at the edge. An appropriate boundary layer solution must be added to satisfy the boundary conditions in these dead weight problems.

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